Consequences of the Cauchy Integral formula

Giacinto Morera

1) Morera Theorem.

Let f be continuous in a region Ω.

Assume that for any $z_0 \in \Omega$ $\exists B(z_0, r) \subset \Omega$ w following property:
if $\partial R \subset B(z_0, r)$ - rectangle then $\oint\limits_{\partial R} f(z)\,dz = 0$.

Then $f \in \mathcal{A}(\Omega)$.

Proof. Fix $z_0 \in \Omega$. We know that $\exists F \in \mathcal{A}$ such that $F'(z) = f(z)$ $\forall z \in B(z_0, r)$. But F differentiable $\Rightarrow$ $f'(z_0)$ exists.
It is true for any $z_0 \Rightarrow f \in \mathcal{A}(\Omega)$ ■

2) Theorem (Cauchy inequality). Ass

$f \in \mathcal{A}(B(z_0, R))$, $|f(z)| \le M$ $\forall z \in B($

Then $|f^{(n)}(z_0)| \le M n! R^{-n}$

Proof. By Cauchy formula for

$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}}$, so $|f^{(n)}(z_0)| \le$

3) 

Joseph Liouville

Theorem (Liouville) Any bounded

is identically constant.

Reminder f is called entire if f

Proof. Let f be bounded $|f$

$p(z) \qquad\qquad z \to \infty \quad |p(z)|$

Thus $\exists R: \left|\frac{1}{p(z)}\right| < 1 \quad \forall z : |z| > R.$

Let $M = \max\limits_{|z| \le R} \frac{1}{|p(z)|}$. Then $\forall z$

So, by Liouville, $\frac{1}{p(z)} \equiv$ const — Co

4) 

<u>Theorem</u> (Weierstras

Let (f_n) be a sequ

in a region Ω. As

Karl Weierstrass

$$\oint_\gamma f\,dz = \lim_{n\to\infty} \oint_\gamma f_n\,dz = 0.$$ By M

2) Fix $z_0 \in \Omega$. Take r: 2

Then $\overline{B(z_0, 2r)} \subset \Omega$,

$f_n \rightrightarrows f$ uniformly on $\overline{B(z_0, r)}$, le

Then, by Cauchy inequalities,

$$|f_n^{(k)}(z) - f^{(k)}(z)| = |(f_n - f)^{(k)}(z)| \leq \frac{k!}{}$$

$f_n^{(k)} \rightrightarrows f^{(k)}$ uniformly on $B(z_0, r)$.

Restatement: $f_n \in A(\Omega)$, $\sum_{n=1}^{\infty} f_n$

Then $f \in A(\Omega)$ and $\forall k$: $\sum_{n=}^{\infty}$